ANALYSIS OF RADIATIVE HEAT TRANSFER IN NONISOTHERMAL CAVITIES

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A simple numerical method is proposed for the calculation of radiative heat transfer in diffuse nonisothermal cavities. Numerical results are given for a parabolic temperature distribution along the length of a cylindrical cavity.

In determining the temperature of solids by means of radiation, it is necessary in many cases to use an open artificial cavity bounded by the surface of a circular cylinder and by a flat bottom. In practice, such a cavity has some kind of nonisothermality. However, even in the most basic work on the theoretical investigation of the radiation characteristics of such cavities [1], it is assumed the cavity walls are at a constant temperature. An attempt is made here based on integral equations [2] to obtain a general expression suitable for numerical computer calculation of the distribution of the effective monochromatic emittance ε_{eff} over the surface of a cavity for arbitrary nonisothermality of the walls.

This study is an extension of previous work [3, 4].

It is assumed the surface of the cavity radiates and reflects diffusely and the polarization of the radiated energy is suppressed in the process of multiple reflection. The temperature of the bottom of the cavity is constant at all points and the temperature field of the lateral surface is axisymmetric. The monochromatic emmissivity of the material from which the cavity is made is independent of temperature. The radiation field is stationary.

As is well known (for example, see [5]), pyrometer radiation readings are proportional to the effective brightness I_{eff} of the sighting point in the sighting direction. In turn, the effective brightness is made up of the brightness of the intrinsic radiation, I_i, and of the reflected radiation, i.e., is determined both by the intrinsic radiation and by the entire family of rays which are incident on the sighting area and which are reflected in the sighting direction (for brevity, the subscript λ for the wave length of the radiation is omitted here and in the following).

Since, under our assumptions, the surface of the cavity emits and reflects diffusely, it is convenient to use not the brightness I but a quantity proportional to it — the surface density of hemispheric radiation E. In setting up the equations, all geometric characteristics are normalized to the radius of the bottom of the cavity.

The basic equations describe the thermal balance for an arbitrarily selected elementary area dF. The elementary strip of area $dF_a(\eta)$ with the coordinate η (the origin of the coordinate system is at the center of the bottom of the cavity) radiates $E_{eff}(\eta)\,dF_a(\eta)$ in all directions. The amount of radiated energy incident on the elementary area $dF(\eta_0)$ with the coordinate η_0 is proportional to the angular coefficient $d\varphi(\eta, \eta_0)$ under which the elementary strip "sees" the elementary area $dF(\eta_0)$ or, taking reversibility into account, $E_{eff}(\eta)\,d\varphi(\eta,\eta_0)\,dF_a(\eta) \equiv E_{eff}(\eta)d\varphi(\eta_0,\eta)dF(\eta_0)$. Then the amount of energy reflected by the elementary area $dF(\eta_0)$ is $R(\eta_0)E_{eff}(\eta)d\varphi(\eta_0,\eta)dF(\eta_0)$. Inclusion of the contribution from all elementary strips of the lateral surface of the cavity is accomplished by integration over the range from 0 to η_L (η_L is the dimensionless depth of the cavity). Similarly, the radiation incident from the bottom surface and reflected by the element $dF(\eta_0)$ is $R(\eta_0)\int_0^1 E_{eff}(\xi)d\varphi(\eta_0, \xi)dF(\eta_0)$, where the angular coefficient $d\varphi(\eta_0,\xi)$ is considered with

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TABLE 1. Dependence of $\varepsilon_{\mathrm{eff},\lambda}(\xi)$ on the Parameters ε_λ, α, T ($\eta_L = 8$, $\lambda = 0.65$ μm)

ε_λ	α	T, °K	ξ: 0	0,2	0,4	0,6	0,8	0,9	0,95	0,97	1,0
	0,1	1000	0,9275	0,9280	0,9298	0,9327	0,9370	0,9397	0,9414	0,9421	0,9436
	0,1	3000	0,9635	0,9638	0,9645	0,9657	0,9674	0,9685	0,9691	0,9694	0,9700
0,5	0	—	0,9880	0,9880	0,9881	0,9882	0,9883	0,9884	0,9884	0,9884	0,9885
			0,9880*								0,9887*
	0,1	1000	0,9479	0,9484	0,9498	0,9522	0,9556	0,9577	0,9590	0,9595	0,9606
	0,1	2000	0,9668	0,9670	0,9678	0,9691	0,9710	0,9722	0,9729	0,9732	0,9738
0,6	0,1	3000	0,9742	0,9744	0,9750	0,9759	0,9772	0,9781	0,9786	0,9788	0,9792
	0	—	0,9918	0,9918	0,9918	0,9919	0,9920	0,9920	0,9920	0,9920	0,9921
	0,3	1000	0,9280	0,9288	0,9313	0,9355	0,9416	0,9456	0,9478	0,9488	0,9506
	0,3	3000	0,9627	0,9631	0,9642	0,9660	0,9686	0,9703	0,9712	0,9717	0,9724
0,7	0,1	1000	0,9637	0,9640	0,9651	0,9669	0,9694	0,9711	0,9720	0,9724	0,9731
	0,1	2000	0,9770	0,9772	0,9778	0,9788	0,9802	0,9811	0,9816	0,9818	0,9822
	0,1	3000	0,9822	0,9824	0,9828	0,9835	0,9845	0,9851	0,9854	0,9857	0,9859
	0	—	0,9945	0,9945	0,9945	0,9945	0,9946	0,9946	0,9946	0,9946	0,9947
	0,3	1000	0,9414	0,9421	0,9442	0,9477	0,9529	0,9562	0,9581	0,9589	0,9603
	0,3	2000	0,9611	0,9615	0,9628	0,9649	0,9679	0,9699	0,9710	0,9714	0,9719
	0,3	3000	0,9699	0,9702	0,9711	0,9726	0,9748	0,9762	0,9770	0,9773	0,9779
0,75	0,1	1000	0,9706	0,9709	0,9718	0,9733	0,9755	0,9768	0,9776	0,9779	0,9785
	0,1	2000	0,9814	0,9816	0,9821	0,9829	0,9841	0,9848	0,9853	0,9854	0,9856
	0,1	3000	0,9857	0,9858	0,9861	0,9867	0,9875	0,9881	0,9884	0,9885	0,9887
	0	—	0,9956	0,9956	0,9956	0,9956	0,9957	0,9957	0,9957	0,9957	0,9957
			0,9956*								0,0961*
	0,3	1000	0,9541	0,9547	0,9564	0,9592	0,9634	0,9661	0,9676	0,9682	0,9693
	0,3	3000	0,9766	0,9768	0,9775	0,9788	0,9805	0,9816	0,9822	0,9825	0,9830
0,8	0,1	1000	0,9771	0,9774	0,9781	0,9793	0,9810	0,9821	0,9827	0,9829	0,9834
	0,1	2000	0,9856	0,9857	0,9861	0,9867	0,9877	0,9883	0,9886	0,9888	0,9889
	0,1	3000	0,9889	0,9890	0,9893	0,9897	0,9904	0,9908	0,9910	0,9911	0,9913
	0	—	0,9965	0,9965	0,9966	0,9966	0,9967	0,9967	0,9967	0,9967	0,9967
	0,5	1000	0,9705	0,9709	0,9721	0,9742	0,9773	0,9792	0,9803	0,9808	0,9816
	0,5	2000	0,9799	0,9802	0,9809	0,9822	0,9841	0,9852	0,9859	0,9862	0,9866
	0,3	3000	0,9842	0,9844	0,9850	0,9859	0,9873	0,9881	0,9886	0,9888	0,9891
	0,3	1000	0,9779	0,9782	0,9790	0,9805	0,9826	0,9839	0,9847	0,9850	0,9855
	0,3	2000	0,9855	0,9857	0,9862	0,9870	0,9883	0,9892	0,9894	0,9896	0,9899
	0,3	3000	0,9888	0,9890	0,9893	0,9899	0,9908	0,9913	0,9916	0,9918	0,9920
0,9	0,2	1000	0,9828	0,9830	0,9836	0,9847	0,9862	0,9872	0,9877	0,9879	0,9883
	0,2	2000	0,9890	0,9891	0,9894	0,9901	0,9909	0,9915	0,9918	0,9919	0,9921
	0,2	3000	0,9916	0,9917	0,9919	0,9923	0,9930	0,9933	0,9935	0,9936	0,9938
	0,1	1000	0,9891	0,9892	0,9896	0,9902	0,9910	0,9916	0,9919	0,9920	0,9922
	0,1	2000	0,9932	0,9932	0,9934	0,9937	0,9942	0,9945	0,9947	0,9947	0,9948
	0,1	3000	0,9947	0,9948	0,9949	0,9951	0,9955	0,9957	0,9958	0,9958	0,9959
	0	—	0,9984	0,9984	0,9984	0,9984	0,9984	0,9984	0,9984	0,9984	0,9984
			0,9984*								0,9986*

*Data from [1].

respect to $dF(\eta_0)$ and the elementary ring of dimensionless radius ξ. Denoting the surface density of effective radiation at the point η_0 by $E_{\mathrm{eff}}(\eta_0)$ and setting up the thermal balance, we obtain

$$E_{\mathrm{eff}}(\eta_0)\,dF(\eta_0) = E_c(\eta_0)\,dF(\eta_0) + R(\eta_0)\Big[dF(\eta_0)\int_0^{\eta_L} E_{\mathrm{eff}}(\eta) \times d\varphi(\eta_0,\ \eta) + dF(\eta_0)\int_0^1 E_{\mathrm{eff}}(\xi)\,d\varphi(\eta_0,\ \xi)\Big] \tag{1}$$

or in dimensionless form after elementary transformations

$$\varepsilon_{\mathrm{eff}}(\eta_0) = \varepsilon(\eta_0) + R(\eta_0)\Big[\int_0^{\eta_L} \varepsilon_{\mathrm{eff}}(\eta)\,f_1(\eta)\,K_1(\eta_0,\ \eta)\,d\eta + f_2(\eta_0)\int_0^1 E_{\mathrm{eff}}(\xi)\,K_2(\eta_0,\ \xi)\,d\xi\Big]. \tag{2}$$

Similarly, for an elementary area on the bottom surface we have

$$\varepsilon_{eff}(\xi_0) = \varepsilon(\xi_0) + R(\xi_0) \int_0^{\eta_L} \varepsilon_{eff}(\eta) f_3(\eta) K_3(\xi_0, \eta)\, d\eta. \tag{3}$$

Values of ε, R, and K and the temperature functions f are given. The effective emissivities ε_{eff} are unknown functions. Thus Eqs. (2) and (3) form a system of two Fredholm integral equations of the second kind in two unknowns. This equation system is valid for monochromatic radiation and also for the case where the emitting surface is "grey."

Since the purpose of the present work is simulation of a black body by a nonisothermal cavity, a rather deep cavity was considered, the dimensionless length of which was $\eta_L = 8$. It was assumed that the temperature fell in the direction of the opening in accordance with the quadratic parabola $T(\eta) = T(0)(1-\alpha(\eta/\eta_L)^2)$. Values of the monochromatic emissivity, of the nonisothermality coefficient $\alpha = (T(0)-T(\eta_L))/T(0)$, and of the temperature of the bottom of the cavity are given in Table 1 together with the calculated data. Formulas for K are given in Appendix A. The temperature functions f were obtained within the limits of applicability of the Wien formula and are given in Appendix B. Since the effective wavelength of optical pyrometers is usually 0.65 μm, the calculations assumed $\lambda = 0.65$ μm and $c_2 = 14380$ μm-deg.

The kernels K of Eqs. (2) and (3) have a weak singularity when $\xi_0 \to 1$, $\eta \to 0$, $\eta_0 \to \eta$ and when $\eta_0 \to 0$, $\xi \to 1$. Unfortunately, this fact was not mentioned in [1], which led to a failure in the numerical solution [6] of such a system of equations in accordance with the scheme given in [1]. To weaken the singularities here, a method was used [7] where the integrand is written in the form of a sum of two functions with one of them containing the "entire" singularity but being exactly integrable and the other without a singularity calculable to any given accuracy by one of the formulas for approximate quadratures. A conversion of the integrals in Eqs. (2) and (3) was carried out in accordance with this. It was found that it was more convenient not to determine the original integrals directly but to calculate their exact values on the basis of the closure principle and reversibility for the angular coefficients φ. The appropriate formulas were obtained by differentiation of the expression M, where M is the angular coefficient between two circles of equal radius located in parallel planes and having a common central normal [8] (the formulas for the angular coefficients and the details of the transformation for reducing the singularities are given in Appendix C). After elementary transformations, the original system of equations is then written for numerical solution as

$$\begin{aligned}\varepsilon_{eff}(\eta_0) = \Big\{\varepsilon(\eta_0) + R(\eta_0)\Big[\int_0^{\eta_L} [\varepsilon_{eff}(\eta) f_1(\eta) - \varepsilon_{eff}(\eta_0)]\, K_1(\eta_0, \eta)\, d\eta \\ + f_2(\eta_0)\Big[\int_0^{1} [\varepsilon_{eff}(\xi) - \varepsilon_{eff}(\xi = 1)]\, K_2(\eta_0, \xi)\, d\xi + \varepsilon_{eff}(\xi = 1) \\ \times \varphi(\eta_0, F_2)\Big]\Big]\Big\} \Big/ [1 - R(\eta_0)\varphi(\eta_0, F_1)],\end{aligned} \tag{2a}$$

$$\begin{aligned}\varepsilon_{eff}(\xi_0) = \varepsilon(\xi_0) + R(\xi_0)\Big\{\int_0^{\eta_L} [f_3(\eta)\varepsilon_{eff}(\eta) - \varepsilon_{eff}(\eta = 0)]\, K_3(\xi_0, \eta)\, d\eta \\ + \varepsilon_{eff}(\eta = 0)\varphi(\xi_0, F_1)\Big\}.\end{aligned} \tag{3a}$$

The calculations were performed by simple numerical iteration. Integration was accomplished by means of the Simpson or Weddle rule; the range from 0 to η_L was divided into 240 equal parts and the range from 0 to 1 into 120 parts.

The calculational scheme was the following: for $\varepsilon_{eff}(\eta) \equiv \varepsilon_{eff}(\xi) \equiv 1$, a table of values for the functions $\varepsilon^{(1)}_{eff}(\eta_0) \equiv \varepsilon_{eff}(\eta)$ was constructed (initial approximation); the resultant set $\varepsilon^{(1)}_{eff}(\eta)$ was substituted in Eq. (3a) and a table was constructed for the functions $\varepsilon^{(1)}_{eff}(\xi_0) \equiv \varepsilon^{(1)}_{eff}(\xi)$ and then the new values of the functions $\varepsilon^{(1)}_{eff}(\eta)$ and $\varepsilon^{(1)}_{eff}(\xi)$ were substituted in Eq. (3a) etc. The iteration process was stopped when the values of $\varepsilon_{eff}(\xi = 0)$ agreed to the fourth significant digit in successive approximations. It must be pointed out that the number of iterations increased from 2-3 for $\varepsilon = 0.9$ to 10-15 for $\varepsilon = 0.5$ and that there was a tendency toward oscillation in the latter case. However, the problem of instability in the computational procedure did not arise for all parameter values assumed.

Calculated values of the effective emissivity $\varepsilon_{eff}(\xi)$ of the bottom of the cavity are given in Table 1 for various parameters $T(\xi) \equiv T(0)$ and α.

The cavity is isothermal for $\alpha = 0$. For this particular case, the data obtained can be compared with previous results [1] which are given in the same table. As follows from the table, the difference in numerical values is 0.0001-0.0002 and is apparently explained by the fact that the algorithm used in this paper, in contrast to that in [1], takes into account the weak singularity of the kernel of the integral equations.

Data for $\varepsilon_{eff}(\xi)$ is not given here for cavities with $\eta_L > 8$ since it has been established that the effect of an increase to a cavity depth greater than 8 yields so slight an improvement of the model of a black body that one can use the tabulated data in practice for $\eta_L > 8$. It must be pointed out that although the calculated data was obtained for $\lambda = 0.65\ \mu m$, it can be used for pyrometric measurements at any radiation wavelength (λ_*). Then the data presented should be normalized to a new temperature $T_*(0)$ calculated from the expression $T_*(0) = 0.65T(0)/\lambda_*$, °K.

The calculations given here were carried out for diffusely emitting and reflecting surfaces. After insignificant modifications (see [3,4]), however, this algorithm can be applied to bodies having diffuse-specular reflection.

APPENDICES A, B, AND C

A. $$d\varphi(\eta_0,\ \eta) = -0{,}5\left(\frac{\partial^2 M}{\partial\eta_i\,\partial\eta_j}\,d\eta_j\right)_{\substack{\eta_i=\eta_0\\ \eta_j=\eta\\ \xi_i=\xi_j=1}} = K_1(\eta_0,\ \eta)\,d\eta;$$

$$K_1(\eta_0,\ \eta) = 0{,}5\left\{1 - |\eta-\eta_0|\,\frac{(\eta-\eta_0)^2+6}{[(\eta-\eta_0)^2+4]^{3/2}}\right\};$$

$$d\varphi(\xi_0,\ \eta) = -0{,}5\left(\frac{1}{\xi_j}\,\frac{\partial^2 M}{\partial\xi_j\,\partial\eta_i}\,d\eta_i\right)_{\substack{\eta_i=\eta\\ \eta_j=0\\ \xi_i=1\\ \xi_j=\xi_0}} = K_3(\xi_0,\ \eta)\,d\eta;$$

$$K_3(\xi_0,\ \eta) = \frac{2(1+\eta^2-\xi_0^2)\,\eta}{[(1+\eta^2+\xi_0^2)^2-4\xi_0^2]^{3/2}};\quad K_2(\eta_0,\ \xi) = K_3(\xi,\ \eta_0)\,\xi;$$

$$M = 0{,}5\left[(\eta_i-\eta_j)^2+\xi_i^2+\xi_j^2-\{[(\eta_i-\eta_j)^2+\xi_i^2+\xi_j^2]^2-4\xi_i^2\xi_j^2\}^{1/2}\right]/\xi_j;$$

B. $$f_1(\eta) = E_{0,\lambda}(\eta)/E_{0,\lambda}(\eta_0);\quad f_2(\eta_0) = E_{0,\lambda}(0)/E_{0,\lambda}(\eta_0);$$

$$f_3(\eta) = E_{0,\lambda}(\eta)/E_{0,\lambda}(0);$$

$$E_{0,\lambda}(\eta) = c_1\exp\left[-c_2/(\lambda T(\eta))\right];\quad T(\eta) = T(0)\left[1-\alpha(\eta/\eta_L)^2\right];$$

C. $$\int_0^{\eta_L} f_1(\eta)\,\varepsilon_{eff}(\eta)\,K_1(\eta_0,\ \eta)\,d\eta = \int_0^{\eta_L}[f_1(\eta)\,\varepsilon_{eff}(\eta)-\varepsilon_{eff}(\eta_0)]$$

$$\times K_1(\eta_0,\ \eta)\,d\eta + \varepsilon_{eff}(\eta_0)\int_0^{\eta_L}K_1(\eta_0,\ \eta)\,d\eta;$$

$$\int_0^1 \varepsilon_{eff}(\xi)\,K_2(\eta_0,\ \xi)\,d\xi = \int_0^1[\varepsilon_{eff}(\xi)-\varepsilon_{eff}(\xi=1)]\,K_2(\eta_0,\ \xi)\,d\xi$$

$$+\varepsilon_{eff}(\xi=1)\int_0^1 K_2(\eta_0,\ \xi)\,d\xi;$$

$$\int_0^{\eta_L}\varepsilon_{eff}(\eta)\,f_3(\eta)\,K_3(\xi_0,\ \eta)\,d\eta = \int_0^{\eta_L}[\varepsilon_{eff}(\eta)\,f_3(\eta)-\varepsilon_{eff}(\eta=0)]$$

$$\times\int_0^{\eta_L}K_3(\xi_0,\ \eta)\,d\eta + \varepsilon_{eff}(\eta=0)\int_0^{\eta_L}K_3(\xi_0,\ \eta)\,d\eta,$$

where

$$\int_0^{\eta_L}K_1(\eta_0,\ \eta)\,d\eta = \varphi(\eta_0,\ F_1)$$

$$= 1-\varphi(\eta_0,\ F_2)-\varphi[(\eta_L-\eta_0),\ F_2'] = 1-0{,}5\left(-\frac{\partial M}{\partial\eta_i}\right)_{\substack{\eta_i=\eta_0\\ \eta_j=0\\ \xi_i=\xi_j=1}}$$

$$-0{,}5\left(-\frac{\partial M}{\partial \eta_i}\right)_{\substack{\eta_i=\eta_L-\eta_0\\ \eta_j=0\\ \xi_i=\xi_j=1}} = 1-0{,}5\left\{\frac{\eta_0^2+2}{(\eta_0^2+4)^{1/2}}\right.$$

$$\left. + \frac{(\eta_L-\eta_0)^2+2}{[(\eta_L-\eta_0)^2+4]^{1/2}} - \eta_L\right\};$$

$$\int\limits_0^1 K_2(\eta_0,\ \xi)\,d\xi = \varphi(\eta_0,\ F_2) = -0{,}5\left(\frac{\partial M}{\partial \eta_i}\right)_{\substack{\eta_i=\eta_0\\ \eta_j=0\\ \xi_i=\xi_j=1}} = 0{,}5\left\{\frac{\eta_0^2+2}{(\eta_0^2+4)^{1/2}} - \eta_0\right\};$$

$$\int\limits_0^{\eta_L} K_3(\xi_0,\ \eta)\,d\eta = 1-\varphi(\xi_0,\ F_2') = 1-0{,}5\left(\frac{1}{\xi_j}\ \frac{\partial M}{\partial \xi_j}\right)_{\substack{\xi_j=\xi_0\\ \xi_i=1\\ \eta_i=\eta_L\\ \eta_j=0}}$$

$$= 0{,}5\left\{1+\frac{\eta_L+\xi_0^2-1}{[(\eta_L^2+\xi_0^2+1)^2-4\xi_0^2]^{1/2}}\right\}.$$

NOTATION

ε, R,	are the radiation and reflection powers of cavity material;
I	is the brightness;
E	is the surface density of semispherical radiation;
F_1, F_2, F'_2,	are the lateral cylindrical surface, bottom surface and fictitious surface;
dF	is the surface element;
η, ξ	are the dimensionless coordinates;
φ	is the angular coefficient;
K	is the kernel of equation;
f	is the temperature function;
α	is the nonisothermity;
c_1, c_2,	are the radiation constants of the Planck law;
λ	is the radiation wavelength;
M	is the angular coefficient between two circles.

Subscripts

eff	is the effective;
c	is the proper;
λ	is the radiation wavelength;
a	is the elementary zone;
0	is the fixed coordinate or blackbody radiation;
L	is the depth of cavity;
1, 2, 3,	are the numbers of kernels and temperature functions;
(n)	is the number of approximations (iterations);
*	is the new value of parameter;
i, j,	are the upper and lower circles, respectively.

LITERATURE CITED

1. E. M. Sparrow, L. V. Albers, and E. R. G. Eckert, Trans ASME, Series C, 84, 73 (1962).
2. G. L. Polyak, in: Convective and Radiative Heat Transfer [in Russian], Izd. Akad. Nauk SSSR (1960).
3. S. P. Rusin, in: Heat and Mass Transfer [in Russian], Vol. 3, Énergiya (1968).
4. S. P. Rusin, "Investigation of the effects of high temperatures on the effective heat conductivity of disperse graphitic carbon materials in a vacuum," Author's abstract of dissertation [in Russian], IVT, Akad. Nauk SSSR (1969).
5. A. A. Poskachei and S. P. Rusin, Temperature Measurements in Electrothermal Devices [in Russian], Énergiya (1967).
6. B. Peavy, J. Res. Nat. Bur. Stand., 70C, No. 2 (1966)
7. V. I. Krylov, Approximate Calculation of Integrals [in Russian], Nauka (1967).
8. M. Jakob, Problems in Heat Transfer [Russian translation], IL (1960)